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MONOCHROMATIC DIRECTED WALKS IN ARC-COLORED DIRECTED GRAPHS

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All graphs considered here are directed and have no loops. For a directed graph D, let $V(D)$ denote the set of vertices of D, $E(D)$ its set of arcs, and $\chi(D)$ the chromatic number of D. D is *symmetric* iff $(x, y)\in E(D)\Leftrightarrow(y, x)\in E(D)$. A directed walk of length k in D is a sequence of k arcs (not necessarily distinct), $e_1, e_2, \ldots, e_k$ such that the initial vertex of e_{i+1} is the terminal vertex of e_i for $i=1, 2, \ldots, k-1$. The directed walk above is called a directed path if all the $k+1$ vertices incident with its arcs are distinct. An arc-coloring of D is a mapping of $E(D)$ into a set C of colors. A subgraph of D is monochromatic if all its arcs have the same color.

Gallai [5] and Roy [7] proved independently the first result connecting the chromatic number of a directed graph with the maximal length of a directed path in it; Every directed graph D contains a directed path of length $\chi(D)-1$. Chvátal [2] noticed that the result of Gallai and Roy implies the following extension of a result of Busolini [1]:

THEOREM A (Chvátal). *Let D be a directed graph and let k, r be positive integers such that $\chi(D)>k^r$; then in any arc-coloring of D with r colors, D contains a monochromatic directed path (and hence a monochromatic directed walk) of length k.*

In view of this theorem, the following two definitions seem natural:

DEFINITION 1. An arc-coloring of a directed graph D is k-free $(k\geqq 2)$ if D does not contain a monochromatic directed walk of length k (i.e., if no directed path of length k and no directed cycle whatsoever is monochromatic).

Define also:

$$C_k(D) = \min\{r\colon \text{there exists a } k\text{-free arc-coloring of } D \text{ with } r \text{ colors}\}.$$

DEFINITION 2. For $k, h\geqq 2$

$$\underline{C}_k(h)=\min\{C_k(D)\colon D \text{ is a directed graph and } \chi(D)=h\},$$

$$\bar{C}_k(h)=\max\{C_k(D)\colon D \text{ is a directed graph and } \chi(D)=h\}.$$

To avoid trivialities we shall consider from now on only directed graphs D for which $\chi(D)\geqq 2$, i.e., $E(D)\neq\emptyset$. By Theorem A, for every directed graph D and every $k\geqq 2$:

$$C_k(D) \geqq \lceil\log_k \chi(D)\rceil, \tag{1}$$

where $\lceil y \rceil$ denotes the smallest integer $\geqq y$, and thus

(2) $$\bar{C}_k(h) \geqq \underline{C}_k(h) \geqq \lceil \log_k h \rceil,$$

for all $k, h \geqq 2$.

In this paper we determine $\bar{C}_k(h)$ exactly for all $k, h \geqq 2$, and show that

$$\lceil \log_k h \rceil = \underline{C}_k(h) \leqq \bar{C}_k(h) \leqq \lceil \log_k h + \log_k \log_k h + 4 \rceil.$$

We also show that if D is symmetric then $C_k(D) = \bar{C}_k(\chi(D))$.

We begin with the following definition:

Definition 3. Let k, r be two positive integers, $k \geqq 2$. Let D be a directed graph, and M a k-free arc-coloring of D with r colors $1, 2, \ldots, r$. For $v \in V(D)$ and $1 \leqq i \leqq r$, denote by $l_M(v, i)$ the maximum length of a monochromatic directed walk of color i beginning at v. ($l_M(v, i) = 0$ if no such walk exists.) With each vertex v of D associate the vector $l_M(v) = (l_M(v, 1), l_M(v, 2), \ldots, l_M(v, r))$. (We shall usually omit the index M, whenever there is no danger of confusion.)

Note that each component of $l_M(v)$ is a nonnegative integer smaller than k, since M is k-free.

The following lemma is a trivial consequence of Definition 3:

Lemma 1. *Let D be a directed graph, and M a k-free arc-coloring of D with r colors* $1, 2, \ldots, r$. *Suppose* $(v, v') \in E(D)$. *If the color of* (v, v') *under* M *is* i, *then:*

$$l_M(v', i) < l_M(v, i).$$

Remark 1. Suppose D is a directed graph, and M is a k-free arc-coloring of D with colors $1, 2, \ldots, r$. If v, v' are adjacent vertices of D, then $l(v) \neq l(v')$, by Lemma 1. Thus, the mapping $v \to l(v)(v \in V(D))$ is a proper vertex-coloring of D, with at most k^r different colors. Thus $\chi(D) \leqq k^r$. This yields a proof of inequalities (1) and (2) which does not depend on Theorem A.

Definition 4. For positive integers k, r let $P(k, r)$ denote the set of all functions b: $\{1, \ldots, r\} \to \{0, \ldots, k-1\}$. If $b, c \in P(k, r)$ write $b \leqq c$ if $b(t) \leqq c(t)$ for all t, $1 \leqq t \leqq r$. Clearly $\leqq$ is a partial order on $P(k, r)$. An $AC(k, r)$ is an antichain in $P(k, r)$, i.e,, a set $F \subset P(k, r)$ such that for every two vectors (=functions) $b, c \in F$ there are indices $1 \leqq s, r \leqq r$ such that $b(s) < c(s)$ and $c(t) < b(t)$.

We denote by $N(k, r)$ the maximal cardinality of an $AC(k, r)$.

E. Sperner [8] proved:

$$N(2, r) = \binom{r}{[r/2]},$$

and De Bruijn, Tenbergen and Kruywijk [3] proved the following generalization of Sperner's result:

Theorem B (De Bruijn, Tenbergen and Kruywijk). *$N(k, r)$ is the number of all vectors $b \in P(k, r)$ satisfying*

$$\sum_{i=1}^{r} b(i) = \left[\frac{1}{2}(k-1)r\right],$$

i.e., $N(k, r)$ is the coefficient of $x^{[(k-1)r/2]}$ in

$$(1+x+x^2+\ldots+x^{k-1})^r.$$

As $(1+x+\ldots+x^{k-1})=(1-x^k)/(1-x)$, it follows easily that

$$N(k, r) = \sum_{i=0}^{[p/k]} (-1)^i \binom{r}{i} \binom{r+p-ki-1}{r-1}, \quad \text{where} \quad p = \left[\frac{1}{2}(k-1)r\right].$$

We shall now establish an upper bound for $C_k(D)$.

LEMMA 2. *For every directed graph D and for every integer $k \geqq 2$:*

$$C_k(D) \leqq \min \{r\colon N(k, r) \geqq \chi(D)\}. \tag{3}$$

PROOF. Given an integer r such that $N(k, r) \geqq \chi(D)$, we shall exhibit a k-free arc-coloring of D with r colors. Choose an $AC(k, r)$ $\{b_1, b_2, \ldots, b_{\chi(D)}\}$ of size $\chi(D)$.

Let $\gamma\colon V(D) \to \{1, \ldots, \chi(D)\}$ be a fixed proper vertex coloring of D. Define an arc-coloring $M\colon E(D) \to \{1, \ldots, r\}$ as follows: If $(v, w) \in E(D)$, let

$$M(v, w) = \min \{t\colon 1 \leqq t \leqq r,\ b_{\gamma(v)}(t) > b_{\gamma(w)}(t)\}.$$

($M(v, w)$ is well defined, since $\gamma(v) \neq \gamma(w)$, and $\{b_1, \ldots, b_{\chi(D)}\}$ is an $AC(k, r)$.)

It remains to show that M is k-free. Suppose $(v_0, v_1), (v_1, v_2), \ldots, (v_{m-1}, v_m)$ is a monochromatic directed walk of length m in D, say, of color i. By the definition of M, $k > b_{\gamma(v_0)}(i) > b_{\gamma(v_1)}(i) > \ldots > b_{\gamma(v_m)}(i) \geqq 0$, and thus $m \leqq k-1$. Therefore M is k-free and (3) follows. □

Combining (1) and Lemma 2 we obtain:

THEOREM 1. *For every directed graph D and every $k \geqq 2$:*

$$\lceil \log_k \chi(D) \rceil \leqq C_k(D) \leqq \min \{r\colon N(k, r) \geqq \chi(D)\}.$$

The next lemma shows that both bounds in Theorem 1 are best possible, and that the upper bound is attained whenever D is symmetric.

LEMMA 3. *Let $h \geqq 2$ be an integer.*

(i) *There exists a directed graph T with $\chi(T)=h$ such that*

$$C_k(T) = \lceil \log_k h \rceil \tag{4}$$

for every $k \geqq 2$.

(ii) *If G is symmetric and $\chi(G) = h$, then*

$$C_k(G) = \min \{r\colon N(k, r) \geqq h\}. \tag{5}$$

for every $k \geqq 2$.

PROOF. (i) Given $h \geqq 2$, let T be a transitive tournament on h vertices, that is: $V(T)=\{v_1, v_2, \ldots, v_h\}$, and $E(T)=\{(v_i, v_j)\colon h \geqq i > j \geqq 1\}$. Obviously $\chi(T)=h$ and thus, by Theorem 1, for every $k \geqq 2$, $C_k(T) \geqq \lceil \log_k h \rceil$. In order to establish (4) we shall exhibit, for every $k \geqq 2$, a k-free arc-coloring of T with r colors, where

$$r = \lceil \log_k h \rceil. \tag{6}$$

Given $k \geqq 2$, define r by (6). Obviously $k^r \geqq h$. For $1 \leqq i \leqq h$, let $b_i \in P(k, r)$ be the k-ary representation of the integer $i-1$, i.e., $b_i = (\alpha_1, \ldots, \alpha_r)$, where $0 \leqq \alpha_\nu < k$ for $1 \leqq \nu \leqq r$ and $i-1 = \sum_{\nu=1}^{r} \alpha_\nu k^{\nu-1}$. (Note that $0 \leqq i-1 < k^r$.) Define a k-free arc-coloring of T with r colors $1, 2, \ldots, r$ exactly as in the proof of Lemma 2:

For $h \geqq i > j \geqq 1$, color the arc (v_i, v_j) of T with color s, where

$$s = \min \{t\colon\ 1 \leqq t \leqq r,\ b_i(t) > b_j(t)\}.$$

(Such an s exists since b_i represents a larger number than b_j.) The argument used in the proof of Lemma 2 shows that this coloring is indeed k-free. This establishes (4).

(ii) Let G be a symmetric directed graph satisfying $\chi(G) = h$. By Theorem 1, for every $k \geqq 2$,

$$C_k(G) \leqq \min \{r\colon\ N(k, r) \geqq h\}.$$

In order to establish (5) we will show that for $k \geqq 2$, if there exists a k-free arc-coloring of G with r colors, then $N(k, r) \geqq h$.

Given $k \geqq 2$, suppose M is a k-free arc-coloring of G with r colors $1, \ldots, r$. By Dilworth's Theorem (see [4]), the partially ordered set $P(k, r)$ is the union of $N(k, r)$ chains $H_1, H_2, \ldots, H_{N(k,r)}$. Define a vertex-coloring $f\colon V(G) \to \{1, 2, \ldots$ $\ldots, N(k, r)\}$ as follows: If $v \in V(G)$ let

$$f(v) = \min \{t\colon\ 1 \leqq t \leqq N(k, r),\ l_M(v) \in H_t\}.$$

Since G is symmetric, Lemma 1 implies that if $(v, w) \in E(G)$ then neither $l_M(v) \leqq l_M(w)$ nor $l_M(w) \leqq l_M(v)$ holds. This means that $f(v) \neq f(w)$ and that f is a proper vertex-coloring of G with $N(k, r)$ colors. Thus $N(k, r) \geqq h$ and (5) follows. □

Combining Lemma 3 and Theorem 1 we obtain:

THEOREM 2. *For every two integers* $k, h \geqq 2$:

$$C_k(h) = \lceil \log_k h \rceil, \tag{7}$$

$$\bar{C}_k(h) = \min \{r\colon\ N(k, r) \geqq h\}. \quad \square \tag{8}$$

REMARK 2. We can prove that there are positive constants c_1, c_2, say $c_1 = 1$ and $c_2 = 4$, such that:

$$\min \{r\colon\ N(k, r) \geqq h\} \leqq \lceil \log_k h + c_1 \log_k \log_k h + c_2 \rceil \tag{9}$$

for every $k, h \geqq 2$.

This shows that $\bar{C}_k(h)$ is not very far from $\underline{C}_k(h)$ and thus the lower and upper bounds for $C_k(D)$, given in Theorem 1, are quite close.

The proof of (9) depends on the trivial estimate

$$N(k, r) \geqq |P(k, r)|/((k-1) \cdot r + 1) \geqq k^{r-1}/r.$$

We omit the detailed proof of (9), since it is rather lengthy and not very complicated.

REMARK 3. It is well known that the problem of deciding whether the chromatic number of a given undirected graph G is greater than 3 is NP-Complete, even under rather severe restrictions on G (see [6, p. 191]). Since $\bar{C}_2(3)=3<4=\bar{C}_2(4)$, part (ii) of Lemma 3 implies that the problem of deciding whether $C_2(D)\leqq 3$ is NP-complete, even if D is a directed symmetric graph.

REMARK 4. Let T be a transitive tournament on h vertices and let G be a complete symmetric directed graph on h vertices. By Lemma 3, for every $k\geqq 2$

$$C_k(T) = \underline{C}_k(h) \leqq \bar{C}_k(h) = C_k(G).$$

Clearly G can be obtained from T by adding $h(h-1)/2$ arcs, one at a time. If H' is obtained from H by adding one arc, then clearly

$$C_k(H) \leqq C_k(H') \leqq C_k(H)+1.$$

It follows that for every $k, h\geqq 2$ and for every m satisfying $\underline{C}_k(h)\leqq m\leqq \bar{C}_k(h)$, there is a directed graph D that satisfies $\chi(D)=h$ and $C_k(D)=m$.

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